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The Divergence of Lagrange Interpolation for $|x|^\alpha$ at Equidistant Nodes

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In 1918 S. N. Bernstein published the surprising result that the sequence of Lagrange interpolation polynomials to $|x|$ at equally spaced nodes in $[-1, 1]$ diverges everywhere, except at zero and the end-points. In the present paper, we prove that the sequence of Lagrange interpolation polynomials corresponding to $|x|^\alpha$ $(0<\alpha\leqslant 1)$ on equidistant nodes in $[-1, 1]$ diverges everywhere in the interval except at zero and the end-points.

1. INTRODUCTION

Consider the infinite triangular matrix $X=\{x_{j,n}\}$, $n=0, 1, 2, ...,$ $0\leqslant j\leqslant n$, where

$$-1\leqslant x_{0,n}<x_{1,n}<\cdots<x_{n,n}\leqslant 1, \qquad (n=0, 1, 2, ...), \tag{1}$$

and denote by $C[-1, 1]$ the Banach space of continuous functions on $[-1, 1]$ equipped with the uniform norm. Then to each $f\in C[-1, 1]$ there corresponds a unique interpolating polynomial $L_n(f, X, .)$ of degree at most n coinciding with f at the nodes of the $(n+1)$th row of X. One of the most important questions in interpolation theory is to characterize under what conditions on f and X the sequence $\{L_n(f, X, x)\}$, $n=0, 1, 2, ...,$ converges to $f(x)$. In 1914 G. Faber [8] discovered the shaking fact that for any matrix X there is a function $f_1\in C[-1, 1]$ such that

$$\lim_{n\to\infty} \|L_n(f_1, X)-f_1\| \neq 0. \tag{2}$$

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Almost twenty years later, in 1931, S. N. Bernstein [3] proved that for every matrix X there is an $f_2 \in C[-1, 1]$ and at least one x, $-1 \leqslant x \leqslant 1$, such that

$$\overline{\lim_{n \to \infty}} |L_n(f_2, X, x)| = \infty. \tag{3}$$

The results of Faber and Bernstein were reinforced by Erdős and Vértesi [7], who proved that for any matrix X there is a function $f_3 \in C[-1, 1]$ such that

$$\overline{\lim_{n \to \infty}} |L_n(f_3, X, x)| = \infty, \qquad \text{for almost all} \quad x \in [-1, 1]. \tag{4}$$

We note that in (4) the word "almost" cannot be deleted. Let us remark that the above mentioned negative results are valid for functions, the construction of which is a difficult process. In 1918 Bernstein [2] proved that for the "most natural" matrix $E = \{-1 + 2j/n\}$, $n \in \mathbb{N}$, $j = 0, 1, ..., n$, and the function $|x|$

$$\overline{\lim_{n \to \infty}} |L_n(|x|, E, x)| = \infty, \qquad \forall x \in (-1, 1), \quad x \neq 0. \tag{5}$$

Bernstein's result suggests that Lagrange interpolation polynomials which are based on equidistant nodes may have very poor approximating properties. For a quantitative result in this direction, see [6].

Motivated by Bernstein's result, in the present paper we consider the behavior of polynomial interpolation for $|x|^\alpha$ $(0 < \alpha \leqslant 1)$ at equidistant nodes. We shall prove that the sequence of polynomial interpolations for $|x|^\alpha$ diverges everywhere in the interval, except at zero and the end-points.

There are at least two reasons for wanting to prove this result. Firstly, it is a new direction in which the result of Bernstein can be generalized and therefore may be able to be combined with generalizations in other directions. Secondly, solving this particular problem contributes to the development of a body of techniques used in studying interpolation polynomials based on equidistant nodes. These techniques are quite different from those used when the interpolation nodes are the zeros of some classical polynomials.

Also it should be pointed out that there are two formulae for expressing Lagrange interpolation polynomials, namely Lagrange's formula and Newton's formula. (See, for example, Blum [4], Chapter 8.2). Lagrange's formula is almost always used for the study of approximating properties in preference to Newton's formula. However, in [2], Bernstein's proof is based on Newton's formula. Also, Brutman and Passow [5], used Newton's representation of the interpolating polynomials in proving the pointwise divergence of Lagrange interpolation for $|x|$ at a broad family of nodes, including the Newman nodes. These facts suggest the problem of proving divergence properties for functions like $|x|$ or $|x|^\alpha$ by using Lagrange's interpolation formula. Thus a second aim of this paper is to establish the result using Lagrange's formula.

2. RESULTS

As indicated above, we shall prove the following result:

THEOREM 1. *Let* $0<\alpha\leqslant 1$ *and* $E=\{-1+2j/n\}$, $n\in\mathbb{N}$, n *even*, $j=0, 1, ..., n$. *Then*

$$\overline{\lim_{n\to\infty}}\, |L_n(|x|^\alpha, E, x)|=\infty, \qquad \forall x\in(-1, 1), \quad x\neq 0.$$

Theorem 1 suggests that the qualitative behavior of the equidistant interpolatory process for $|x|^\alpha$ does not depend on α.

3. PREREQUISITES FOR THE PROOF

We begin with the introduction of the Pochhammer notation $(.)_j$, the generalized hypergeometric function ${}_pF_q$ and some important properties. For $a\in\mathbb{R}$, $j=0, 1, 2, ...,$ we define $(a)_j$ by

$$(a)_0=1,$$

$$(a)_j=a(a+1)\cdots(a+j-1), \qquad (j=1, 2, 3, ...).$$

The generalized hypergeometric function is introduced by

$${}_pF_q\left(\begin{matrix}\alpha_1, a_2, ..., \alpha_p\\ \beta_1, \beta_2, ..., \beta_q\end{matrix}\,\middle|\, z\right)=\sum_{j=0}^{\infty}\frac{(\alpha_1)_j\,(\alpha_2)_j\cdots(\alpha_p)_j}{(\beta_1)_j\,(\beta_2)_j\cdots(\beta_q)_j}\frac{z^j}{j!}.$$

By a standard calculation one can simply establish the following results:

LEMMA 2. *Let* $m\in\mathbb{N}$, $k=1, 2, ..., m$, $j=0, 1, 2, ...$ *and* $0\leqslant x\leqslant 1$. *Then*

(a) $$\frac{1}{j+k+mx}=\frac{1}{k+mx}\frac{(k+mx)_j}{(k+1+mx)_j}.$$

(b) $(m+k+j)!=(m+k)!\,(m+k+1)_j$.

(c) $(m-k-j)! = \dfrac{(-1)^j\,(m-k)!}{(k-m)_j}, \qquad (j \leqslant m-k).$

(d) $(k-m)_j = 0, \qquad (j = m+1-k,\ m+2-k,\ \ldots).$

LEMMA 3. *We will denote the gamma function by* $\Gamma(.)$.

(a) *Let* $s = d+e-a-b-c$ *and* $s \neq 0$. *Then*

$$ {}_3F_2\left(\begin{matrix} a,\ b,\ c \\ d,\ e \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(d)\,\Gamma(e)\,\Gamma(s)}{\Gamma(a)\,\Gamma(s+b)\,\Gamma(s+c)}\ {}_3F_2\left(\begin{matrix} d-a,\ e-a,\ s \\ s+b,\ s+c \end{matrix}\,\middle|\, 1\right). $$

(b) *Let* $c \neq 0, -1, -2, \ldots$ *and* $c-a-b > 0$. *Then*

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}. $$

(c) *Let* $m \in \mathbb{N}$, $k = 1, 2, \ldots, m$. *Further denote by*

$$ a_j(x) = \binom{2m}{m+j} \frac{1}{j+mx}, \qquad (j = 1, 2, \ldots, m,\ 0 \leqslant x \leqslant 1). $$

Then

$$ \sum_{j=k}^{m} (-1)^{j-1}\, a_j(x) $$
$$ = (-1)^{k+1}\, a_k(x)\ {}_3F_2\left(\begin{matrix} k+mx,\ k-m,\ 1 \\ k+1+mx,\ 1+k+m \end{matrix}\,\middle|\, 1\right). $$

(d) *For* $m \in \mathbb{N}$, $k = 1, 2, \ldots, m$, $0 \leqslant x \leqslant 1$ *we have*

$$ {}_3F_2\left(\begin{matrix} k+mx,\ k-m,\ 1 \\ k+1+mx,\ 1+k+m \end{matrix}\,\middle|\, 1\right) > \tfrac{1}{2}. $$

(e) *For* $m \in \mathbb{N}$, $k = 1, 2, \ldots, m$, $0 < x \leqslant 1$ *we have*

$$ {}_3F_2\left(\begin{matrix} k+mx,\ k-m,\ 1 \\ k+1+mx,\ 1+k+m \end{matrix}\,\middle|\, 1\right) < \frac{k+m}{1+2m}\,\frac{k+mx}{k+mx-1}. $$

Proof. (a) See p. 104 of [10].

(b) See, for example, p. 556 of [1].

(c) By using Lemma 2(a–d), the result can be directly established.

(d) By Lemma 3(a) one gets

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} k+mx, k-m, 1 \\ k+1+mx, 1+k+m \end{matrix}\,\middle|\, 1\right) \\
&\quad = \sum_{j=0}^{\infty} \frac{(1+m-mx)_j}{(1+k+m)_j} \frac{(k+mx)}{(1+2m+j)} \\
&\quad > \frac{1}{2} \frac{(k+mx)}{(1+k+m)}\, {}_2F_1\left(\begin{matrix} 1, 1+m-mx \\ 2+k+m \end{matrix}\,\middle|\, 1\right) = \frac{1}{2},
\end{aligned}
$$

where we have used Lemma 3(b) and the obvious fact that $1+2m+j < 2(1+k+m+j)$.

(e) Again, by Lemma 3(a), we have

$$
\begin{aligned}
{}_3F_2\left(\begin{matrix} k+mx, k-m, 1 \\ k+1+mx, 1+k+m \end{matrix}\,\middle|\, 1\right) &= \sum_{j=0}^{\infty} \frac{(1+m-mx)_j}{(1+k+m)_j} \frac{(k+mx)}{(1+2m+j)} \\
&< \frac{k+mx}{1+2m}\, {}_2F_1\left(\begin{matrix} 1, 1+m-mx \\ 1+k+m \end{matrix}\,\middle|\, 1\right).
\end{aligned}
$$

Employing Lemma 3(b) to the last expression establishes the result. ∎

Next we formulate some properties of a certain function which are important in the proof of the main result.

LEMMA 4. *Let* $0 < \alpha \leqslant 1$ *and*

$$
g_\alpha(y) = \frac{1-y^\alpha}{1-y}, \qquad y \in (0, \infty).
$$

Then we have

$$
\begin{aligned}
&\text{(a)} \quad g_\alpha(y) \geqslant 0 \qquad \textit{and} \qquad g_\alpha(1) := \alpha. \\
&\text{(b)} \quad g'_\alpha(y) \leqslant 0 \qquad \textit{and} \qquad g'_\alpha(1) := -\frac{\alpha(1-\alpha)}{2}. \\
&\text{(c)} \quad g''_\alpha(y) \geqslant 0 \qquad \textit{and} \qquad g''_\alpha(1) := \frac{\alpha(1-\alpha)(2-\alpha)}{3}.
\end{aligned}
$$

For the restricted case $y \in (0, 1]$, a proof of these facts is given in [11]. This proof can directly be extended to $y \in (0, \infty)$. Note also that Lemma 4 becomes false for $\alpha > 1$. Now we study a certain inequality which plays a crucial role in the proof of Theorem 1.

LEMMA 5. *Let $m \in \mathbb{N}$, $0 < \alpha \leqslant 1$, $0 < x \leqslant 1$. Then one has*

$$\sum_{j=1}^{m} (-1)^{j-1} \binom{2m}{m+j} \frac{j^{\alpha} - (mx)^{\alpha}}{j^2 - (mx)^2}$$
$$\leqslant \frac{1}{(mx)^{1-\alpha}} \sum_{j=1}^{m} (-1)^{j-1} \binom{2m}{m+j} \frac{1}{j+mx}. \tag{6}$$

Note that for $\alpha = 1$ the inequality becomes sharp for all $m \in \mathbb{N}$, $0 < x \leqslant 1$, whereas for $\alpha > 1$ the inequality is generally not true.

Proof. Let g_α be defined as in Lemma 4. Then Lemma 5 is equivalent to showing that

$$S_m^{\alpha}(x) := \sum_{j=1}^{m} (-1)^{j-1} \binom{2m}{m+j} \frac{1}{j+mx} \left(1 - g_\alpha\left(\frac{j}{mx}\right)\right) \geqslant 0 \tag{7}$$

for $m \in \mathbb{N}$, $0 < \alpha \leqslant 1$ and $0 < x \leqslant 1$. We restrict the discussion to the even case since the odd case can be established by similar arguments. Thus let $m \in \mathbb{N}$, m even. By the definition of $a_j(x)$ (see Lemma 3(c)) we rewrite $S_m^{\alpha}(x)$ from (7) more concisely as

$$\sum_{\substack{j=1 \\ j \text{ odd}}}^{m} \left[a_j(x) \int_0^{j/(mx)} - g'_\alpha(z)\, dz - a_{j+1}(x) \int_0^{(j+1)/(mx)} - g'_\alpha(z)\, dz \right].$$

Now we change the order of summation and write $S_m^{\alpha}(x)$ in the following form:

$$\left[\sum_{j=1}^{m} (-1)^{j-1} a_j(x) \right] \int_0^{1/(mx)} - g'_\alpha(z)\, dz - \left[\sum_{j=2}^{m} (-1)^{j} a_j(x) \right] \int_{1/(mx)}^{2/(mx)} - g'_\alpha(z)\, dz$$
$$+ \left[\sum_{j=3}^{m} (-1)^{j-1} a_j(x) \right] \int_{2/(mx)}^{3/(mx)} - g'_\alpha(z)\, dz$$
$$- \left[\sum_{j=4}^{m} (-1)^{j} a_j(x) \right] \int_{3/(mx)}^{4/(mx)} - g'_\alpha(z)\, dz$$
$$+ \cdots + \left[\sum_{j=m-1}^{m} (-1)^{j-1} a_j(x) \right] \int_{(m-2)/(mx)}^{(m-1)/(mx)} - g'_\alpha(z)\, dz$$
$$- \left[\sum_{j=m}^{m} (-1)^{j} a_j(x) \right] \int_{(m-1)/(mx)}^{m/(mx)} - g'_\alpha(z)\, dz.$$

Recall from Lemma 4 that $-g'_\alpha$ is a non negative and decreasing function and that each of the sums $\sum_{j=k}^{m} (-1)^{j-1} a_j(x) \geqslant 0$ for k odd. Thus

$$\begin{aligned} S_m^\alpha(x) \geqslant & \left[2 \sum_{j=1}^{m} (-1)^{j-1} a_j(x) - a_1(x) \right] \int_{1/(mx)}^{2/(mx)} - g'_\alpha(z)\, dz \\ & + \left[2 \sum_{j=3}^{m} (-1)^{j-1} a_j(x) - a_3(x) \right] \int_{3/(mx)}^{4/(mx)} - g'_\alpha(z)\, dz + \cdots \\ & + \left[2 \sum_{j=m-1}^{m} (-1)^{j-1} a_j(x) - a_{m-1}(x) \right] \int_{(m-1)/(mx)}^{m/(mx)} - g'_\alpha(z)\, dz. \end{aligned}$$

Finally we make use of Lemma 3(c) for $k = 1, 3, 5, ..., m-1$ combined with Lemma 3(d) to establish the desired result.

4. PROOF OF THEOREM 1

Let $m \in \mathbb{N}$, $n = 2m$, $0 < \alpha \leqslant 1$ and $0 < x < 1$. Further denote by

$$\begin{aligned} x_j = x_{j,n} &= -1 + \frac{2j}{n}, \qquad (j = 0, 1, ..., n), \\ w(x) &= (x - x_0) \cdots (x - x_n), \\ l_k(x) &= \frac{w(x)}{(x - x_k)\, w'(x_k)}, \qquad (k = 0, 1, ..., n), \\ f_\alpha(x) &= |x|^\alpha, \\ L_n(f_\alpha, x) &= L_n(f_\alpha, E, x). \end{aligned}$$

Clearly, since $-1, 0, 1$ are interpolation nodes for all even n, and since f_α and $L_{2m}(f_\alpha)$ are even functions, we can restrict ourselves to the interval $(0, 1)$. We begin with Lagrange's formula.

$$L_{2m}(f_\alpha, x) = \sum_{j=0}^{2m} f_\alpha(x_j)\, l_j(x) = \sum_{j=0}^{2m} f_\alpha(x_j) \frac{w(x)}{(x - x_j)\, w'(x_j)}. \tag{8}$$

A standard calculation establishes

$$w'(x_{2m-j}) = w'(x_j) = (-1)^j \frac{(2m)!}{m^{2m} \binom{2m}{j}}, \tag{9}$$

and

$$w(x) = \frac{(-1)^m}{m} \frac{1}{m^{2m}} \frac{\sin \pi m x}{\pi} \Gamma(1+m(1+x))\, \Gamma(1+m(1-x)). \qquad (10)$$

Combining (8), (9) and (10) we get (after changing the order of summation)

$$L_{2m}(f_\alpha, x) = -\frac{\sin \pi m x}{\pi} \frac{\Gamma(1+m(1+x))\, \Gamma(1+m(1-x))}{(2m)!} \times 2mx \sum_{j=1}^{m} (-1)^j \binom{2m}{m+j} \frac{f_\alpha\left(\frac{j}{m}\right)}{j^2-(mx)^2}. \qquad (11)$$

Now we interpolate the function $h(x) \equiv 1$, and we establish

$$1 = -\frac{\sin \pi m x}{\pi} \frac{\Gamma(1+m(1+x))\, \Gamma(1+m(1-x))}{(2m)!} \times 2mx \left[\sum_{j=1}^{m} (-1)^j \binom{2m}{m+j} \frac{1}{j^2-(mx)^2} - \frac{1}{2m^2} \binom{2m}{m} \frac{1}{x^2} \right]. \qquad (12)$$

Multiplying (12) by x^α and combining with (11), we establish for $x \in (0, 1)$ and all $\alpha > 0$

$$|L_{2m}(|x|^\alpha, x) - |x|^\alpha| = \left| \frac{\sin \pi m x}{\pi} \right| \cdot \frac{\Gamma(1+m(1+x))\, \Gamma(1+m(1-x))}{(2m)!} \times m^{1-\alpha} x \left| \frac{1}{(mx)^{2-\alpha}} \binom{2m}{m} - 2 \sum_{j=1}^{m} (-1)^{j-1} \binom{2m}{m+j} \frac{j^\alpha - (mx)^\alpha}{j^2-(mx)^2} \right|. \qquad (13)$$

At this stage we cannot see the "divergence", because the term with the gamma functions is decreasing to zero as m tends to infinity and moreover, we have no information on the size of the third term. We combine Lemma 5 and Lemma 3(c, e) to estimate (for $0 < \alpha \leqslant 1$)

$$\begin{aligned}
&\sum_{j=1}^{m} (-1)^{j-1} \binom{2m}{m+j} \frac{j^{\alpha}-(mx)^{\alpha}}{j^2-(mx)^2} \\
&\qquad \leqslant \frac{1}{(mx)^{1-\alpha}} \sum_{j=1}^{m} (-1)^{j-1} \binom{2m}{m+j} \frac{1}{j+mx} \\
&\qquad = \frac{1}{(mx)^{1-\alpha}} \binom{2m}{m+1} \frac{1}{1+mx} \, {}_3F_2\left(\left. \begin{matrix} 1+mx,\ 1-m,\ 1 \\ 2+mx,\ 2+m \end{matrix} \right| 1 \right) \\
&\qquad < \frac{1}{(mx)^{2-\alpha}} \binom{2m}{m} \frac{m}{1+2m}. \qquad (14)
\end{aligned}$$

Combining (13) and (14) we establish the following lower bound:

$$\begin{aligned}
|L_{2m}(|x|^{\alpha}, x) - |x|^{\alpha}| &\geqslant \left| \frac{\sin \pi m x}{\pi} \right| \cdot \frac{\Gamma(1+m(1+x))\, \Gamma(1+m(1-x))}{(\Gamma(1+m))^2\, m(1+2m)} \\
&= A_m(x) \cdot B_m(x). \qquad (15)
\end{aligned}$$

To complete the proof, it will be sufficient to show that $B_m(x) \to \infty$ as $m \to \infty$ for all fixed $x \in (0, 1)$, while there is an increasing subsequence of indices m, such that for an appropriate small $\varepsilon > 0$ we have $A_m(x) \geqslant \varepsilon > 0$.

We begin with $B_m(x)$. To this end we employ the asymptotic expansion of $\log \Gamma(x)$ (see, for example, p. 257 of [1]).

$$\log \Gamma(x) = \left(x - \frac{1}{2} \right) \log x - x + \frac{1}{2} \log 2\pi + O\left(\frac{1}{x} \right), \qquad \text{as} \quad x \to \infty.$$

We consider

$$\begin{aligned}
\frac{1}{m} \log B_m(x) = \frac{1}{m} \log \Gamma(1+m(1+x)) + \frac{1}{m} \log \Gamma(1+m(1-x)) \\
- \frac{2}{m} \log \Gamma(m+1) + O\left(\frac{\log m}{m} \right).
\end{aligned}$$

Then as $m \to \infty$ we obtain

$$\begin{aligned}
\frac{1}{m} \log B_m(x) = &\frac{\frac{1}{2} + m(1+x)}{m} \log(1+m(1+x)) \\
&+ \frac{\frac{1}{2} + m(1-x)}{m} \log(1+m(1-x)) \\
&- \frac{2m+1}{m} \log(m+1) + O\left(\frac{\log m}{m} \right)
\end{aligned}$$

$$= \frac{\frac{1}{2} + m(1+x)}{m} \log\left(\frac{1+m(1+x)}{m+1}\right)$$

$$+ \frac{\frac{1}{2} + m(1-x)}{m} \log\left(\frac{1+m(1-x)}{m+1}\right) + O\left(\frac{\log m}{m}\right).$$

Consequently we get

$$\lim_{m\to\infty} \frac{1}{m} \log B_m(x) = (1+x)\log(1+x) + (1-x)\log(1-x) > 0,$$

$$\forall x \in (0, 1).$$

So $B_m(x)$ diverges with exponential order. It remains to show that $A_m(x)$ is bounded away from zero infinitely often. We consider two cases:

Case A ($x \in \mathbb{Q}$). We write x in the form a/b with $a, b \in \mathbb{N}$ and $(a, b) = 1$. It is easy to give an explicit increasing subsequence $\{m_j\}_{j \geqslant 0}$ of indices such that (for an appropriate small $\varepsilon > 0$) one has

$$\left| \sin \pi m_j \frac{a}{b} \right| = \varepsilon > 0, \qquad (\varepsilon = \varepsilon(x),\ j = 0, 1, ...). \tag{16}$$

For instance, the subsequence $\{jb+1\}_{j \geqslant 0}$ will work.

Case B ($x \in \mathbb{R} \backslash \mathbb{Q}$). Then the sequence $\{|\sin \pi m x|\}_{m \geqslant 1}$ is dense in $[0, 1]$ (this follows from the continuity of sin and the well known fact that the Kronecker sequence $\{mx \bmod 1\}_{m \geqslant 1}$ is dense in $[0, 1]$. Of course, the Kronecker sequence is uniformly distributed mod 1 (see [9]). Thus we can always find an increasing subsequence of indices such that, for some small positive ε, $|\sin \pi m_j x| \geqslant \varepsilon(x) > 0$ for all $j = 0, 1, ...$.

This completes the proof of the theorem.

5. OPEN PROBLEMS

We close with stating some (interesting) open problems. (We will refer to the function f_α defined above.)

(a) Motivated by numerical computations, we conjecture that the divergence of the sequence $\{L_n(f_\alpha, E, x)\}_{n \geqslant 1}$ at $x \in (-1, 1)$, $x \neq 0$ takes place for all $\alpha > 0$ (except α an even integer).

(b) We mention that we have proved the divergence for the subsequence of polynomials of even degree while the odd degree case is still open. So we formulate the following problem. Is it true that (for $0 < \alpha \leqslant 1$)

$$\overline{\lim_{n \to \infty}} |L_{2n-1}(|x|^{\alpha}, E, x)| = \infty, \qquad \forall x \in (-1, 1), \qquad x \neq 0?$$

(c) Do the Lagrange interpolatory parabolas for $|x|^{\alpha}$ reflect their divergence (convergence) behavior solely by the function $|x|$? More precisely, we ask, whether there is a matrix X and at least one $\alpha \in (0, 1)$ (or $\alpha \in \mathbb{R}_+ \backslash 2\mathbb{N}$ and $\alpha \neq 1$) such that

$$\overline{\lim_{n \to \infty}} \|L_n(f_{\alpha}, X) - f_{\alpha}\| \neq \overline{\lim_{n \to \infty}} \|L_n(f_1, X) - f_1\|,$$

or not?

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